On a Condition that a Trigonometrical Series should have a Certain Form.

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§ 1. In a recent communication to the Society I have illustrated the fact that the derived series of the Fourier series of functions of bounded variation play a definite part in the theory of Fourier series. Some of the more interesting theorems in that theory can only be stated in all their generality when the coefficients of such derived series take the place of the Fourier constants of a function. I have also recently shown that Lebesgue's theorem, whether in its original or in its extended form, with regard to the usual convergence of a Fourier series when summed in the Cesàro manner is equally true for the derived series of Fourier series of functions of bounded variation. I have also pointed out that, in considering the effect of all known convergence factors in producing usual convergence, it is immaterial whether the series considered be a Fourier series, or such a derived series.

We are thus led to regard the derived series of the Fourier series of functions of bounded variation as a kind of pseudo-Fourier series, possessing properties which are identical with or analogous to those of Fourier series, properly so-called. In particular we are led to ask ourselves what is the necessary and sufficient condition that a trigonometrical series should have the form in question. One answer is of course immediate. The integrated series must converge to a function of bounded variation. This is merely a statement in slightly different language of the property in question. We require a condition of a simpler formal character, one which does not require us to solve the difficult problem as to whether an assigned trigonometrical series not only converges but also has for sum a function of bounded variation.

I have already indicated the corresponding answer in the case when the trigonometrical series is required to be the Fourier series of a function of a particular class. If the class be the class of functions whose $(1+p)$th power is summable, this condition is that

$$\int_{-\pi}^{\pi} |f_n(x)|^{1+p} dx$$

should be a bounded function of n, where $f_n(x)$ denotes the nth Cesàro partial summation of the series considered.

In the present communication I propose to show that the necessary and sufficient condition that a trigonometrical series should be the derived series of the Fourier series of a function of bounded variation is that

$$\int_{-\pi}^{\pi} |f_n(x)|\,dx$$

should be a bounded function of n.

This result seems to me to be worthy of the attention of the Society for various reasons. It not only illustrates the fact that a function, about which all that we know is that it is summable, is necessarily more than merely summable, but it of itself justifies our regarding such derived series as having their definite place in our theory. Moreover, the method by which it is obtained has an interest of its own. It involves the consideration of bounded successions of integrals of positive functions, and we are led by the exigencies of the reasoning to conclude *a priori* the probability of such successions of integrals containing sequences. It was, in fact, in this way that I was led to remark that this result is immediately deducible from reasoning which I had already employed. We have, in fact, the following theorem:—If the integrands of a bounded succession of integrals are bounded below in their ensemble, there is in every sub-succession of the succession of the integrals a sequence which converges to a lower semi-continuous upper semi-integral, in other words, to an asymmetrically continuous function of bounded variation of a certain type.

The cases in which we can assert that an oscillating succession contains a sequence are very few in number. Almost the only one known is that discovered by Arzelà, and which, in the extended and modified form given to it by myself, requires the condition of uniform and homogeneous oscillation on one side at least. Such successions have come to have a considerable theoretical importance in the abstract theory of sets, as well as in the applications to the theory of functions of a real variable.

It is noteworthy that in the present instance, though this does not come out in the proof, there is uniform and homogeneous oscillation on the left. This follows indeed from a fact that I have long ago signalised, viz., that the non-uniformity of the oscillation of a succession of monotone continuous functions is always visible.

With regard to the main result of the paper it will be noted that it gives us at the same time the necessary and sufficient condition that a trigonometrical series should be the Fourier series of a function of bounded variation.

It may finally be remarked that if instead of expressing the Fourier coefficients of the Fourier series as ordinary integrals involving the function

itself in the integrand, we express them in terms of integrals with respect to the indefinite integral of that function, the form thus given to the Fourier series is identical with the corresponding one for the derived series of a Fourier series, the only difference being that the function with respect to which the integration is to be performed is in the latter case a function of bounded variation which is not in general an integral.

§ 2. We first prove the theorem with respect to successions of integrals to which reference has been made.

Theorem.—*If a succession of integrals of functions which are bounded below (above) in their ensemble oscillates boundedly, there is in every sub-succession a sequence of the integrals, converging to a lower (upper) semi-continuous upper (lower) semi-integral.*

It will be sufficient to prove the former of the two alternative statements in the theorem. Since the succession of integrals is bounded and that of the integrands $f_n(x)$ is bounded below, the latter succession is semi-integrable below; therefore all the upper functions and all the lower functions of the succession of integrals are upper semi-integrals.*

Again, from the fact that the succession $f_n(x)$ is bounded below, it follows that $\int_E f_n(x)\,dx$ has no negative double limit, as $n \to \infty$ and $E \to 0$, and accordingly that the succession of integrals oscillates uniformly and homogeneously below.† Hence all the upper and all the lower functions of the succession of integrals are lower semi-continuous functions.

Now an upper semi-integral is the sum of an integral, which is a continuous function, and a monotone increasing function, which, in our case, is accordingly a lower semi-continuous function, and therefore continuous on the left.

But I have elsewhere‡ proved that, if all the upper, or all the lower, functions of a succession are continuous on one side at least, the same at each point for all such upper or lower limiting functions, then a sequence of the functions can be found having an unique limiting function. In our case the functions are the integrals $\int f_n(x)\,dx$, so that, by this theorem, we can find a succession of integers $n_1, n_2, \ldots$, such that $\int f_{n_i}(x)\,dx$ converges, as $i \to \infty$, to an unique limiting function, which, being one of the lower and upper limiting functions of the succession of integrals, is, by what has been

* W. H. Young, "Semi-integrals and Oscillating Successions of Functions," 'Lond. Math. Soc. Proc.,' 1910, Ser. 2, vol. 9, pp. 300–301, §§ 15–16.

† W. H. Young, "Successions of Integrals and Fourier Series," *ibid.*, 1912, vol. 11, p. 51.

‡ W. H. Young, "On Homogeneous Oscillation of Successions of Functions," *ibid.* 1912, vol. 8, p. 356, Cor. 4.

pointed out, a lower semi-continuous upper semi-integral, and therefore continuous on the left.

Since the succession $f_n(x)$ may equally well be any sub-succession of the given succession, this proves the theorem in the case when the integrands are bounded below in their ensemble. Similarly the alternative case may be proved.

§ 3. We are now able to obtain the result which forms the main object of this communication.

Theorem.—*The necessary and sufficient condition for a trigonometrical series*

$$\sum_{m=1} (a_m \cos mx + b_m \sin mx) \tag{1}$$

to be the derived series of the Fourier series of a function of bounded variation is that

$$\int_{-\pi}^{\pi} | f_n(x) | \, dx \leq \mathrm{C},$$

where C *is a finite constant independent of* n, *and* $f_n(x)$ *is the Cesàro partial summation of the series* (1).

First, to prove the sufficiency of the condition.

Since $\int_{-\pi}^{x} |f_n(x)| \, dx$ is a bounded function of (n, x), the same is true of $\int_{-\pi}^{x} \{ |f_n(x)| + f_n(x) \} \, dx$. The integrands of these two integrals being positive, we can apply to each of them the theorem of § 2. Thus we can find such a succession of integers n_1', n_2', ..., that, for this succession of values of n, the first of the integrals describes a sequence. The corresponding values of the second integral form a sub-succession to which we again apply the theorem, and find a succession of integers n_1, n_2, ..., from among n_1', n_2', ..., so that as n describes these values, the second integral describes a sequence. Therefore as n describes the succession n_1, n_2, ..., both the integrals describe sequences, and therefore their difference, namely $\int_{-\pi}^{x} f_n(x) \, dx$ also describes a sequence. By § 2, the limiting functions of the two first sequences are semi-integrals, and therefore functions of bounded variation. Hence the limiting function of the last sequence, say $g(x)$, is a function of bounded variation, and we have

$$g(x) = \underset{r \to \infty}{\mathrm{Lt}} \int_{-\pi}^{x} f_{n_r}(x) \, dx = \underset{r \to \infty}{\mathrm{Lt}} \, \mathrm{F}_{n_r}(x), \text{ say,} \tag{2}$$

where
$$\mathrm{F}_{n_r}(x) = \int_{-\pi}^{x} \left\{ \sum_{i=1}^{i=n_r} \left(1 - \frac{i-1}{n_r}\right) (a_i \cos ix + b_i \sin ix) \right\}$$

$$= \sum_{i=1}^{i=n_r} \left(1 - \frac{i-1}{n_r}\right) (a_i \sin ix - b_i \cos ix - b_i)/i. \tag{3}$$

Since $F_{n_r}(x)$ is a bounded function of (n, x), we may integrate (2) term by term after multiplying both sides by $\cos mx$. Thus

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(x) \cos mx \, dx = \operatorname*{Lt}_{r\to\infty}\left(-1+\frac{m-1}{n_r}\right) b_m/m = -b_m/m. \tag{4}$$

Similarly, multiplying (2) by $\sin mx$ and integrating term by term

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(x) \sin mx \, dx = a_m/m. \tag{5}$$

From (4) and (5),

$$g(x) \sim \text{const.} + \sum_{m=1} \{-b_m \cos mx + a_m \sin mx\}/m.$$

This shows that our trigonometrical series (1) is the derived series of the Fourier series of the function $g(x)$ of bounded variation, provided the given condition is satisfied. The condition is therefore sufficient. Next to prove that it is necessary.

Let $g(x)$ be the function of bounded variation, corresponding to which the series (1) is the derived series of the Fourier series. Since $g(x)$ is the difference of two monotone increasing functions, and therefore

$$f_n(x) = f_{n,1}(x) - f_{n,2}(x),$$

where $f_{n,1}(x)$ and $f_{n,2}(x)$ are the Cesàro partial summations of the derived series of the Fourier series of these monotone increasing functions, it is only necessary to prove the necessity of the condition when the function of bounded variation is a monotone increasing function.

Now, when $g(x)$ is a monotone increasing function, $f_n(x)$ is positive, for

$$f_n(x) = \frac{1}{2n\pi}\int_{-\pi}^{\pi} \frac{\sin^2 \frac{1}{2}n(x-t)}{\sin^2 \frac{1}{2}(x-t)} \, dg(t).$$

Therefore, in this case $\int_{-\pi}^{\pi} |f_n(x)| \, dx$ is $\int_{-\pi}^{\pi} f_n(x) \, dx$, and is given by the right-hand side of (3). Denoting by $T_n(x)$ the Cesàro partial summation of the Fourier series of which our series is the derived series, we have, therefore,

$$\int_{-\pi}^{\pi} f_n(x) \, dx = T_n(x) - T_n(-\pi) \leq C,$$

where C is the upper bound of the Cesàro partial summations of the Fourier series of $g(x)$, which, as is known,* converges boundedly in the Cesàro manner, since $g(x)$ has bounded variation. Thus the condition is necessary when $g(x)$ is monotone increasing, and therefore also, in the general case, when $g(x)$ is a function of bounded variation.

* W. H. Young, "On the Integration of Fourier Series," 'Lond. Math. Soc. Proc.,' 1910, Ser. 2, vol. 9, pp. 452–453, § 3.

§ 4. Bearing in mind what has been said in § 1, it will be seen that the following theorem completes the set of tests of the type considered:—

Theorem.—*The necessary and sufficient condition that a given trigonometrical series should be the Fourier series of a bounded function is that* $|f_n(x)| \leq \mathrm{B}$ *for all values of* n *and* x, B *being a finite constant.*

That this condition is necessary is evident from mere inspection of the usual expression for $f_n(x)$. That it is sufficient follows from reasoning of a similar but simpler character to that employed in the analogous theorems.

In fact, if $f_n(x)$ is a bounded function of (n, x), the integrated series necessarily oscillates uniformly and homogeneously when summed in the Cesàro manner, index unity. Accordingly, a sequence of these Cesàro partial summations can be found converging to an integral, since $\int_{\mathrm{E}} f_n(x)$ has the unique double limit zero when $\mathrm{E} \to 0$, $n \to \infty$. In other words, we have

$$\int_a^x f(x)\,dx = \underset{r\to\infty}{\mathrm{Lt}} \int_a^x f_{n_r}(x)\,dx,$$

where, moreover, $f(x)$ is a certain bounded function.* Multiplying both sides of this equation by $\cos mx$, or by $\sin mx$, and integrating term by term, we see that the integrated series is a Fourier series, having $\mathrm{F}(x)$ for corresponding function; that is, the integrated series is the Fourier series of the integral of a bounded function, whence the theorem follows.

* "Successions of Integrals and Fourier Series," *loc. cit.*, p. 31.